magic. The mediæval sorcerers undoubtedly drew much of their ritual from astrological sources, although the use of circles is not necessarily derived directly therefrom.[8]

These references should suffice to establish the connection between the oriented circle and the universe, and it only remains to show that the circle was knowingly employed in this sense, to completely prove the thesis.

In the text-books of mediæval magic there will frequently be found instructions to invoke from each quarter of the compass, or again to call certain spirits from a given direction. Such rules occur in the *Clavicula,* but in the absence of references the author cannot recollect the locus, nor can he give the names of other books although such instructions certainly appear in them.

The practice of the "eastward position" in churches, however, is alone sufficient to show that there is a traditional association of ideas of the kind sought. The practice of ceremonial processions with the Sun, such as is frequently to be observed in Catholic services, is an additional demonstration. If, however, we proceed further we shall only be retracing the ground which has been already covered by students of heliolatry.

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NOTES ON PANDIAGONAL AND ASSOCIATED MAGIC SQUARES.

The reader's attention is invited to the plan of a magic square of the thirteenth order shown in Fig. 1 which is original with the writer. It is composed of four magic squares of the fourth order, two of the fifth order, two of the seventh order, two of the ninth order, one of the eleventh order and finally the total square of the thirteenth order, thus making twelve perfect magics in one, several of which have cell numbers in common with each other.

To construct this square it became necessary to take the arithmetical series 1, 2, 3.... 169 and resolve it into different series capable of making the sub-squares. A close study of the constitution of all these squares became a prerequisite, and the following observations are in a large part the fruit of the effort to accomplish the square shown. This article is intended however to cover more particularly the constitution of squares of the fifth

[8] Note a mention of magic circles in Cicero, *De Divinatione.*

order. The results naturally apply in a large degree to all magic squares, but especially to those of uneven orders.

It has of course been long known that magic squares can be built with series other than the natural series 1, 2, 3.... n^2, but the perplexing fact was discovered, that although a magic square might result from one set of numbers when arranged by some rule, yet when put together by another method the construction would fail to give magic results, although the second rule would work all right with another series. It therefore became apparent that these rules were in a way only *accidentally right.* With the view of explaining

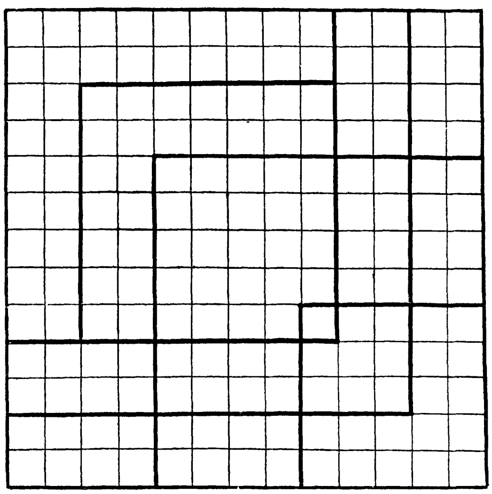

Fig. 1.

these puzzling facts, we will endeavor to analyze the magic square and discover, if possible, its *raison d'être.*

The simplest, and therefore what may be termed a "primitive" square, is one in which a single number is so disposed that every column contains this number once and only once. Such a square is shown in Fig. 2, which is only one of many other arrangements by which the same result will follow. In this square every column has the same summation (a) and it is therefore, in a limited sense, a magic square.

Our next observation is that the empty cells of this figure may be filled with other quantities, resulting, under proper arrangement, in a square whose every column will still have a constant summation. Such a square is shown in Fig. 3 in which every column sums

$a + b + c + d + g$, each quantity appearing once and only once in each row, column, and diagonal. These squares however have the fatal defect of duplicate numbers, which can not be tolerated. This defect can be removed by constructing another primitive square, of five other numbers (Fig. 4), superimposing one square upon the

a				
		a		
				a
	a			
			a	

Fig. 2.

a	b	c	d	g
d	g	a	b	c
b	c	d	g	a
g	a	b	c	d
c	d	g	a	b

Fig. 3.

x	y	s	t	v
s	t	v	x	y
v	x	y	s	t
y	s	t	v	x
t	v	x	y	s

Fig. 4.

other, and adding together the numbers thus brought together. This idea is De la Hire's theory, and it lies at the very foundation of magical science. If however we add a to x in one cell and in another cell add them together again, duplicate numbers will still result, but this can be obviated by making the geometrical pattern in one square the *reverse* of the same pattern in the other square. This idea is illustrated in Figs. 3 and 4, wherein the positions of a and v are reversed. Hence, in the addition of cell numbers in two such squares a series of diverse numbers must result. These series are necessarily magical because the resulting square is so. We can now lay down the first law regarding the constitution of magical series, viz., *A magic series is made by the addition, term to term, of x quantities to x other quantities.*

As an example, let us take five quantities, a, b, c, d and g, and add them successively to five other quantities x, y, s, t and v, and we have the series:

$$\begin{array}{ccccc}
a+x & a+y & a+s & a+t & a+v \\
b+x & b+y & b+s & b+t & b+v \\
c+x & c+y & c+s & c+t & c+v \\
d+x & d+y & d+s & d+t & d+v \\
g+x & g+y & g+s & g+t & g+v
\end{array}$$

This series, with *any values* given to the respective symbols, will produce magic squares if properly arranged. It is therefore a *universal series,* being convertible into any other possible series.

We will now study this series, to discover its peculiar properties if we can, so that hereafter it may be possible at a glance to

determine whether or not a given set of values can produce magical results. First, there will be found in this series a property which may be laid down as a law, viz.:

There is a constant difference between the homologous numbers of any two rows or columns, whether adjacent to each other or not. For example, between the members of the first row and the corresponding members of the second row there is always the constant difference of $a - b$. Also between the third and fourth rows there is a constant difference $c - d$, and between the second and third columns we find the constant difference $y - s$ etc., etc. Second, it will be seen that any column can occupy any vertical position in the system and that any row could exchange place with any other row. (As any column could therefore occupy any of five positions in the system, in the arrangement of columns we see a total of

$$5 \times 4 \times 3 \times 2 \times 1 = 120 \text{ choices.}$$

Also we see a choice of 120 in the rows, and these two factors indicate a total of 14,400 different arrangements of the 25 numbers and a similar number of variants in the resulting squares, to which point we will revert later on.)

This *uniformity of difference* between homologous numbers of any two rows, or columns, appears to be the *only essential* quality of a magical series. It will be further seen that this must necessarily be so, because of the process by which the series is made, i. e., the successive addition of the terms of one series to those of the other series.

As the next step we will take two series of five numbers each, and, with these quantities we will construct the square shown in Fig. 5 which combines the two primitives, Figs. 3 and 4.

By observation we see that this is a "pure" square, i. e., in no row, column, or diagonal is any quantity *repeated* or *lacking*. Because any value may be assigned to each of the ten symbols used, it will be seen that this species of square depends for its peculiar properties *upon the geometrical arrangement of its members and not on their arithmetical values*; also that the five numbers represented by the symbols a, b, c, d, g, need not bear any special ratio to each other, and the same heterogeneity may obtain between the numbers represented by x, y, s, t, v.

There is however another species of magic square which is termed "associated" or "regular," and which has the property that the sum of any two diametrically opposite numbers equals twice

the contents of the central cell. If we suppose Fig. 5 to be such a square we at once obtain the following equations:

$$
\begin{array}{ll}
(1) & (d+s)+(d+x)=2d+2y \therefore x+s=2y \\
(2) & (d+t)+(d+v)=2d+2y \therefore t+v=2y \\
(3) & (c+y)+(g+y)=2d+2y \therefore c+g=2d \\
(4) & (a+y)+(b+y)=2d+2y \therefore a+b=2d
\end{array}
$$

Hence it is evident that if we are to have an associated square, the element d must be an arithmetical mean between the quantities c and g and also between a and b. Also, y must be a mean between x and s, and between t and v. It therefore follows that an associated square can only be made when the proper *arithmetical relations* exist between the numbers used, while the construction of a continuous or pandiagonal square depends upon the *method of arrangement* of the numbers.

a x	b y	c s	d t	g v
d s	g t	a v	b x	c y
b v	c x	d y	g s	a t
g y	a s	b t	c v	d x
c t	d v	g x	a y	b s

Fig. 5.

1	23	137	223	263
167	229	191	7	53
197	37	83	173	157
89	101	163	227	67
193	257	73	17	107

Fig. 6.

163	257	1	53	173
227	73	23	167	157
67	17	137	229	197
89	107	223	191	37
101	193	267	7	83

Fig. 7.

The proper relations are embraced in the above outline, i. e., that the *central term of each of the five (or x) quantities shall be a mean between the diametrically opposite pair.* For example, 1.4.9.14.17, or 1.2.3.4.5, or 1.2.10.18.19, or 1.10.11.12.21 are all series which, when combined with similar series, will yield magical series from which associated magic squares may be constructed.

The failure to appreciate this distinction between pandiagonal and associated squares is responsible for much confusion that exists, and because the natural series $1.2.3.4\ldots.n^2$ *happens,* as it were, accidentally to be such a series as will yield associated squares, *empirical rules have been evolved for the production of squares which are only applicable to such a series,* and which consequently fail when another series is used. For example, the old time Indian rule of regular diagonal progression when applied to a *certain class* of series will yield magic results, but when applied to another class of series it fails utterly!

As an example in point, the following series, which is composed of prime numbers, will yield the continuous or nasik magic square shown in Fig. 6, but a square made from the same numbers arranged according to the old Indian rule is not magic in its diagonals as shown in Fig. 7.

1	7	37	67	73
17	23	53	83	89
101	107	137	167	173
157	163	193	223	229
191	197	227	257	263

The fundamentally *partial* rules, given by some authors, have elevated the *central row* of the proposed numbers into a sort of axis on which they propose to build. This central row of the series is thrown by their rules into one or the other diagonal of the completed square. The fact that this central row adds to the correct summation is, as before stated, simply an accident accruing to the normal series. The central row does *not* sum correctly in many magical series, and rules which throw this row into a diagonal are therefore incompetent to take care of such series.

Returning to the general square, Fig. 5, it will be seen that because each row, column and diagonal contains every one of the ten quantities composing the series, the sum of these ten quantities equals the summation of the square. Hence it is easy to make a square whose summation shall be any desired amount, and also at the same time to make the square contain certain predetermined numbers.

For example, suppose it is desired to make a square whose summation shall be 666, and which shall likewise contain the numbers 6, 111, 3 and 222. To solve this problem, two sets of five numbers each must be selected, the sum of the two sets being 666, and the sums of some members in pairs being the special numbers wished. The two series of five numbers each in this case may be

$$
\begin{array}{rcrcl}
3 & & 0 & & \\
6 & & 108 & & \\
20 & & 216 & & \\
50 & & 100 & & \\
100 & & 63 & & \\
\hline
179 & + & 487 & = & 666
\end{array}
$$

from which by regular process we derive the magic square series

3	6	20	50	100
111	114	128	158	208
219	222	236	266	316
103	106	120	150	200
66	69	83	113	163

containing the four predetermined numbers. The resulting magic square is shown in Fig. 8, the summation of which is 666 and which is continuous or pandiagonal. As many as eight predetermined numbers can be made to appear together with a predetermined summation, in a square of the fifth order, but in this case duplicate numbers can hardly be avoided if the numbers are selected at random.

3	114	236	150	163
266	200	66	6	128
69	20	158	316	103
208	219	106	83	50
120	113	100	111	222

Fig. 8.

1	59	8	15	19
14	12	13	21	42
33	4	48	11	6
45	5	26	16	10
9	22	7	39	25

Fig. 9.

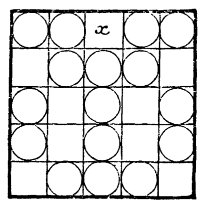

Fig. 10.

We may go still further and force four predetermined numbers into four certain cells of any chosen column or row as per following example:

A certain person was born on the 1st day of the 8th month, was married at the age of 19, had 15 children and is now 102 years old. Make a pandiagonal square whose $S = 102$ and in which numbers 1, 8, 15, 19 shall occupy the first, third, fourth and fifth cells of the upper row.

Referring to the universal square given in Fig. 5,

$$\begin{array}{ll} \text{Let } a = 0 & x = 1 \\ c = 3 & s = 5 \\ d = 9 & t = 6 \\ g = 6 & v = 13 \end{array}$$

These eight quantities sum 43, so that the other pair (b and y) must sum 59, $(43 + 59 = 102)$. Making therefore $b = 20$ and $y = 39$, and replacing these values in Fig. 5, we get the desired square shown in Fig. 9.

As previously shown, continuous squares are dependent on the geometrical placing of the numbers, while associated squares depend also upon the arithmetical qualities of the numbers used. In this connection it may be of interest to note that *a square of third order can not be made continuous,* but *must be associated*; a square of the fourth order may be made *either continuous or associated, but can not combine these qualities*; in a square of the fifth order *both qualities may belong to the same square.* As shown in my article in *The Monist* for July, 1909, very many continuous or nasik squares of the fifth order may be constructed, and it will now be proven that associated nasik squares of this order can only be made in fewer numbers.

In a continuous or "pure" square each number of the sub-series must appear once and only once in each row, column, and diagonal (broken or entire). Drawing a square, Fig. 10, and placing in it

t	v	x	y	s
x	y	s	t	v
s	t	v	x	y
v	x	y	s	t
y	s	t	v	x

Fig. 11.

a	b	c	d	g
d	g	a	b	c
b	c	d	g	a
g	a	b	c	d
c	d	g	a	b

Fig. 12.

1	5	2	3	4
3	4	$\boxed{1}$	5	2
5	$\boxed{2}$	3	4	1
4	1	5	2	3
2	3	4	1	5

Fig. 13.

an element x as shown, the cells in which this element *can not* then be placed are marked with circles. In the second row only two cells are found vacant, thus giving only two choices, indicating two forms of the square. Drawing now another square, Fig. 11, and filling its first row with five numbers, represented by the symbols t, v, x, y and s, and choosing one of the two permissible cells for x in the second row, it will be seen that there can be but *two* variants when once the first row is filled, the contents of every cell in the square being forced as soon as the choice between the two cells in the second row is made for x. For the other subsidiary square, Fig. 12, with numbers represented by the symbols a, b, c, d and g, there is *no choice,* except in the filling of the first row. If this row is filled, for example, as shown in Fig. 12, all the other cells in this square *must* be filled in the manner shown in order that it may fit Fig. 11.

Now, therefore, taking the five symbols x, y, s, t, v, any one of them may be placed in the first cell of the first line of Fig. 11.

For the second cell there will remain a choice of four symbols, for the third cell three, for the fourth cell two, for the fifth cell no choice, and finally in the second line there will be a choice of two cells. In the second subsidiary there will be, as before, a choice of five, four, three and finally two, and no choice in the second row. Collecting these choices we have $(5\times4\times3\times2\times2) \times (5\times4\times3\times2) = 28{,}800$, so that exactly 28,800 continuous or nasik squares of the fifth order may be made from any series derived from ten numbers. Only one-eighth of these, or 3600, will be really diverse since any square shows eight manifestations by turning and reflection.

The question now arises, how many of these 3600 diverse nasik squares are also associated? To determine this query, let us take the regular series 1.2.3....25 made from the ten numbers

1	1	3	4	5
0	5	10	15	20

Making the first subsidiary square with the numbers 1.2.3.4.5, (Fig. 13) as the square is to be associated, the central cell must contain the number 3. Selecting the upward left-hand diagonal to work on, we can place either 1, 2, 4 or 5 in the next upward cell of this diagonal (a choice of four). Choosing 4, we *must* then write 2 in its associated cell. For the upper corner cell there remains a choice of two numbers, 1 and 5. Selecting 1, the location of 5 is forced. Next, by inspection it will be seen that the number 1 may be placed in either of the cells marked □, giving two choices. Selecting the upper cell, every remaining cell in the square becomes *forced.* For this square we have therefore only

$$4\times2\times2 = 16 \text{ choices.}$$

For the second subsidiary square Fig. 14 the number 10 must occupy the central cell. In the left-hand upper diagonal adjacent cell we can place either 0, 5, 15 or 20 (four choices). Selecting 0 for this cell, 20 becomes fixed in the cell associated with that containing 0. In the upper left-hand corner cell we can place either 5 or 15 (two choices). Selecting 15, 5 becomes fixed. Now we can not in this square have any further choices, because all other 15's *must* be located as shown, and so with all the rest of the numbers, as may be easily verified. The total number of choices in this square are therefore $4\times2 = 8$, and for both of the two subsidiaries, $16\times8 = 128$. Furthermore, as we have seen that each square has eight manifestations, there are really only $^{128}/_{8} = 16$ *different plans*

of squares of this order which combine the associated and nasik features.

If a continuous square is expanded indefinitely, any square block of twenty-five figures will be magic. Hence, with any given square, twenty-five squares may be made, only one of which can be associated. There are therefore $16 \times 25 = 400$ variants which can be made according to the above plan. We have however just now shown that there are 3600 different plans of continuous squares of this order. Hence it is seen that only one plan in nine ($3600/400 = 9$) of continuous squares can be made *associated* by shifting the lines and columns. Bearing in mind the fact that eight variants of a square may be made by turning and reflection, it is interesting to note that if we wish a square of the fifth order to be both associated and continuous, we can locate unity in any one of the four cells marked □ in Fig. 15, but by no constructive process can the de-

15	10	5	0	20
5	0	20	15	10
20	15	10	5	0
10	5	0	20	15
0	20	15	10	5

Fig. 14.

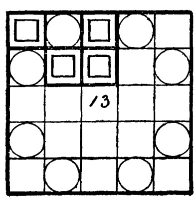

Fig. 15.

1	47	6	43	5	48
35	17	30	21	31	16
36	12	41	8	40	13
7	45	2	49	3	44
29	19	34	15	33	20
42	10	37	14	38	9

Fig. 16.

sired result be effected, if unity is located in any cells marked ○. Then having selected the cell for 1, the cell next to 1 in the same column with the central cell (13) must contain one of the four numbers 7, 9, 17, or 19. The choices thus entailed yield our estimated number of sixteen diverse associated nasik squares, which may be naturally increased eight times by turning and reflection.

That we must place in the same row with 1 and 13, one of the four numbers 7, 9, 17, or 19 is apparent when it is noted that of the series

1	2	3	4	5
0	5	10	15	20

having placed 3 and 10 in the central cells of the two subsidiaries, and 0 and 1 in two other cells, we are then compelled to use in the same line either 5 or 15 in one subsidiary and either 2 or 4 in the

other subsidiary, the combination of which four numbers affords only 7 and 17, or 9 and 19.

With these facts now before us we are better prepared to construct such squares as in which only prime numbers are used, etc. Reviewing a list of primes it will be seen that every number excepting 2 and 5 ends in either 1, 3, 7 or 9. Arranging them therefore in regular order according to their terminal figures as

1	11	31	41	
3	13	23	43	
7	17	37	47	etc.

we can make an easier selection of desired numbers.

A little trial develops the fact that it is impossible to make five rows of prime numbers, showing the same differences between every row, or members thereof, and therefore a *set* of differences must be found, such as 6, 30, 30, 6 (or some other suitable *set*). Using the above set of differences, the series of twenty-five primes

157	13	23	147	109	31	111	138	36	66	102	100	72
145	25	17	153	61	139	59	32	134	104	68	98	70
16	154	144	26	57	56	30	112	136	99	105	60	110
22	148	156	14	113	114	140	58	34	65	71	133	37
97	73	94	76	151	18	21	89	146	135	35	29	141
79	91	78	92	27	82	150	155	11	63	107	33	137
74	96	75	95	143	159	15	20	88	115	55	101	69
90	80	93	77	19	24	81	149	152	54	116	103	67
164	6	3	167	85	142	158	12	28	64	106	108	62
7	163	168	86	1	132	44	39	125	50	48	118	124
162	8	84	2	169	38	126	131	45	120	122	52	46
5	83	161	10	166	129	43	40	128	123	117	49	51
87	165	9	160	4	41	127	130	42	47	53	121	119

Fig. 17.

shown on page 146 may be found. In this series it will be seen that similar differences exist between the homologous numbers of any row, or column, and it is therefore only necessary to arrange the numbers by a regular rule, in order to produce the magic square in Fig. 6.

These facts throw a flood of light upon a problem on which

gallons of ink have been wasted, i. e., the production of pandiagonal and regular squares of the sixth order. It is impossible to distribute six marks among the thirty-six cells of this square so that one and only one mark shall appear in every column, row and diagonal. Hence a *primitive* pandiagonal magic square of this order is excluded by a geometrical necessity. In this case the natural series of numbers is not adapted to construct pandiagonal squares of this order. That the difficulty is simply an arithmetical one is proven by the fact that 6×6 pandiagonal squares can be made with *other series,* as shown in Fig. 16. We are indebted to Dr. C. Planck for this interesting square which is magic in its six rows, six columns and twelve diagonals, and is also four-ply and nine-ply, i. e., any square group of four or nine cells respectively, sums four or nine times the mean. It is constructed from a series made by arranging the numbers 1 to 49 in a square and eliminating all numbers in the central line and column, thus leaving thirty-six numbers as follows:

1	2	3	5	6	7
8	9	10	12	13	14
15	16	17	19	20	21
29	30	31	33	34	35
36	37	38	40	41	42
43	44	45	47	48	49

Fig. 17 shows the completed square which is illustrated in skeleton form in Fig. 1. All the sub-squares are faultless except the small internal 3×3, in which one diagonal is incorrect.

FRIERSON, LA. L. S. FRIERSON.

TWO MORE FORMS OF MAGIC SQUARES.

SERRATED MAGIC SQUARES.

The curious form of magic squares, which is to be described here, is a style possessing a striking difference from the general form of magic squares.

To conform with the saw-tooth edges of this class of squares, I have ventured to call them "serrated" magic squares.

A square containing the series 1, 2, 3, 4, ... 41 is shown in Fig. 1. Its diagonals are the horizontal and vertical series of nine numbers, as A in Fig. 2. Its rows and columns are zigzag as